1. [5+5+5=15 **Points**]

Consider the families of one-dimensional time-continuous systems $x' = f_a(x)$ depending on a parameter $a \in \mathbb{R}$ with $f_a(x)$ defined as below. For each system, sketch the bifurcation diagram including sketches of representative one-dimensional phase portraits and classify the bifurcations of equilibrium points. If your student number is $\mathsf{s}\,\mathsf{n}_1\mathsf{n}_2\mathsf{n}_3\mathsf{n}_4\mathsf{n}_5\mathsf{n}_6\mathsf{n}_7$ then

(a) $f_a(x) = x^3 + (-1)^{\mathsf{n}_7} ax$,

(b) $f_a(x) = x^4 - x^2 + (-1)^{\mathsf{n}_6} a$,

(c) $f_a(x) = \sin x + (-1)^{\mathsf{n}_5} ax$.

You need to upload your handwritten answer before 9:10 (extra time students before 9:15)

2. [2+7+8=15 Points]
Consider the planar system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & a \\ -1 & -b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where a and b are positive constants.

(a) Show from the eigenvalues that the equilibrium at the origin is asymptotically stable.

(b) Identify the regions in the relevant portion of the first quadrant of the $a-b$ plane where the system has similar phase portraits. For each region, determine the canonical form and sketch the phase portrait.

(c) Use Lasalle's Invariance Principle to prove again that the equilibrium at the origin is asymptotically stable and that the basin of attraction is the full plane. (Hint: first construct a suitable Lyapuonov function of arbitrary radius centered at the origin.)

You need to upload your handwritten answer before 9:45 (extra time students before 9:55)

3. **[15 Points]**

A solution $X(t)$ of a time-continuous dynamical system $X' = F(X)$ for a C^1 vector field $F : \mathbb{R}^k \to \mathbb{R}^k$ is called *recurrent* if $X(t_n) \to X(0)$ as $n \to \infty$ for some sequence of times (t_n) with $t_n \to \infty$ as $n \to \infty$. Prove that a gradient dynamical system has no non-constant recurrent solutions.

You need to upload your handwritten answer before 10:20 (extra time students before 10:35)

4. **[15 Points]**

For a family of C^∞ functions $f_\lambda : [0,1] \to [0,1]$, $\lambda \in \mathbb{R}$, consider the discrete-time dynamical systems $x_{n+1} = f_\lambda(x_n)$, $n \in \mathbb{Z}_{\geq 0}$. Suppose that f_{λ_0} has fixed point at x_0 with $|f'(x_0)| > 1$. Show that there is an interval I containing x_0 and a interval J containing λ_0 such that for $\lambda \in J$, f_λ has a unique fixed point in I which is repelling (i.e. a source) and no other orbits that lie entirely in I.

You need to upload your handwritten answer before 10:55 (extra time students before 11:15)

5. [**15 Points**]

For a C^∞ map $f : \mathbb{R}^2 \to \mathbb{R}$, consider the family of discrete-time systems $x_{n+1} = f(x_n, a)$ with parameter $a \in \mathbb{R}$. Suppose that for $x_0, a_0 \in \mathbb{R}$,

(i) $f(x_0, a_0) = x_0$,

(ii) $\frac{\partial f}{\partial x}(x_0, a_0) = 1$,

(iii) $\frac{\partial^2 f}{\partial x^2}(x_0, a_0) \neq 0$, and

(iv) $\frac{\partial f}{\partial a}(x_0, a_0) \neq 0$.

Show that the discrete-time system has a saddle-node bifurcation at (x_0, a_0).

You need to upload your handwritten answer before 11:30 (extra time students before 11:55)

6. [**15 Points**]

For two continuous functions $f, g : [0,1] \to [0,1]$, consider the discrete-time dynamical systems $x_{n+1} = f(x_n)$ and $y_{n+1} = g(y_n)$, $n \in \mathbb{Z}_{\geq 0}$. Suppose that these systems are semi-conjugate, i.e. there is a function $h : [0,1] \to [0,1]$ such that $h \circ f = g \circ h$ where h is continuous, surjective and every point in $[0,1]$ has under h finitely many pre-images in $[0,1]$. Show that if the dynamical system defined via f is chaotic then the dynamical system defined via g is chaotic.

You need to upload your handwritten answer before 12:05 (extra time students before 12:35)